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# Generalized pure spinors

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**Abstract**

With every non-zero spinor $x$ there is associated a totally isotropic subspace $N(x)$ of the underlying vector space $W$; the subspace $N(x)$ consists of all vectors annihilating the spinor. The dimension $\nu$ of $N(x)$—the *nullity* of $x$—is an invariant of the action of the Clifford group and provides a coarse classification of spinors. According to a terminology introduced by Cartan and Chevalley, a spinor is *pure* if the space $N(x)$ is maximal among totally isotropic subspaces of $W$. In this paper, we consider 'partially pure' spinors, i.e. Weyl (= semi-) spinors such that $0 < \nu < n$, where $2n$ is the dimension of the vector space $W$ endowed with a neutral quadratic form. All homogeneous polynomial invariants are shown to vanish on Weyl spinors of positive nullity. We also show that there are no Weyl spinors of nullity $\nu$ such that $n-4 < \nu < n$ or $\nu = n-5$. We compute the dimensions of spaces of partially pure spinors and show that, for $n = 4$ or $> 5$, generic spinors have nullity 0. The paper contains also a heuristic introduction to the notion of pure spinors, comments about their geometrical and physical significance and remarks on the history of the subject.

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## 1. Introduction and historical remarks

### 1.1. Notation and heuristic considerations

This paper is intended to be self-contained; for this reason we present, in considerable detail, not only our notation and terminology, but also some of the standard definitions and results in algebra, needed to prove the theorems on partially pure spinors. Some of the well-known and important results are formulated as *Propositions*, with references to the literature instead of proofs. We use the label *Theorem* when we feel there is need for a proof, because either the result is new or no suitable reference is known to us.

*1.1.1. Vector spaces and quadratic forms*

There are two 'natural' quadratic forms, associated with vector spaces of low dimension: the *determinant* and the *Pfaffian*; they lead to the 'generic' isomorphisms among the classical groups and their Lie algebras [D,W2]. They are also useful in describing spinors associated with such spaces.

If $K$ is an Abelian group and 0 denotes its neutral element, then $K^\times = K \setminus \{0\}$. In particular, if $K$ is a field, then $K^\times$ is its multiplicative group. Throughout this paper, $W$ denotes a finite-dimensional vector space over a field $K$ of characteristic $\neq 2$. Beginning with §2.6 we assume $K = \mathbb{C}$. Let $W'$ be the *dual* of $W$, i.e. the vector space of all $K$-linear maps from $W$ to $K$. The value of the 1-form $w'$ on $w \in W$ is often denoted by $\langle w, w' \rangle$. If $V \subset W$, then $V^\circ \subset W'$ is the vector space of all forms vanishing on all elements of $V$. If $V$ is a vector subspace of $W$, then $V^{\circ\circ} = V$. If $h$ is a non-degenerate quadratic form on $W$, then the pair $(W, h)$ is said to be a *quadratic space*; there is the associated isomorphism $g : W \to W'$ obtained by 'polarization' of $h$, viz.

$$2\langle w_1, g(w_2) \rangle = h(w_1 + w_2) - h(w_1) - h(w_2), \quad \text{where} \quad w_1, w_2 \in W. \tag{1}$$

It satisfies $\langle w, g(w) \rangle = h(w)$ and $g(w_1, w_2) = \langle w_1, g(w_2) \rangle$ is the *scalar product* of the vectors $w_1$ and $w_2$. If $V \subset W$, then $V^\perp$ is the vector subspace of $W$ consisting of all vectors orthogonal to all elements of $V$. A vector subspace $V$ of $W$ is said to be *isotropic* (null[2]) if $h$ restricted to $V$ is degenerate; it is *totally isotropic* if $h|V = 0$, or, equivalently, if $V \subset V^\perp$. The *index* of $h$ is the dimension of maximal totally isotropic (*mti*) subspaces of $W$. For $h$ defined on a space of dimension $m = 2n$ or $2n + 1$, the largest value of the index is

[2] According to F. Klein (see Ch. 4 in [K]), the adjective *isotropic* was used by A. Ribaucour (1845–1893) in the context of the geometry of $\mathbb{C}^2$: the vector $(1, \mathrm{i})$ is 'isotropic' because a rotation by the angle $\phi$ in the complex plane maps it into $e^{\mathrm{i}\phi}(1, \mathrm{i})$, a vector parallel to $(1, \mathrm{i})$. This observation does not generalize to higher dimensions; physicists often use the phrase 'null elements', which, however, does not translate well into French and is not recognized by pure mathematicians. Cartan [C1] used the expression 'optical direction', which seems appropriate, but has not gained acceptance.

$n$. A quadratic form of maximal index on an even-dimensional vector space is said to be *neutral*. If $f : V_1 \to V_2$ is a $K$-linear map of vector spaces, then the *transposed* map ${}^t f : V_2' \to V_1'$ is defined by $\langle f(v_1), v_2' \rangle = \langle v_1, {}^t f(v_2') \rangle$, where $v_1 \in V_1$ and $v_2' \in V_2'$. In particular, $g$ is *symmetric*, ${}^t g = g$. We identify, in the usual manner, $V''$ with $V$.

*1.1.2. The Grassmann algebra*

Throughout this paper, by an algebra we mean an associative algebra with a unit element, which is denoted by 1. A homomorphism of algebras is understood to map one unit into another. We denote by $\wedge V$ the Grassmann algebra of $V$. A linear map $f : V_1 \to V_2$ extends to the homomorphism of algebras $\wedge f : \wedge V_1 \to \wedge V_2$. The pairing $V \times V' \to K$ is extended to $\wedge V \times \wedge V' \to K$ so that $\langle v_1 \wedge \cdots \wedge v_k, v_1' \wedge \cdots \wedge v_k' \rangle = \det(\langle v_i, v_j' \rangle)$, where $v_i \in V$ and $v_j' \in V'$ for $i, j = 1, \ldots, k$ and $k = 1, \ldots, \dim V$. The Grassmann algebra has two important involutive maps: the *$\mathbb{Z}_2$-grading automorphism* $\alpha$ such that $\alpha(1) = 1$, $\alpha(v) = -v$, for $v \in V$, and the *main antiautomorphism* $\beta$ such that $\beta(1) = 1$, $\beta(v) = v$; if $x \in \wedge^k V$, then $\beta(x) = (-1)^{k(k-1)/2} x$. The even and odd subspaces of $\wedge V$ are denoted by $\wedge^+ V$ and $\wedge^- V$, respectively, and there is the decomposition

$$\wedge V = \wedge^+ V \oplus \wedge^- V. \tag{2}$$

If $x \in \wedge V$, then $e(x) : \wedge V \to \wedge V$ is a linear map, the *exterior product* by $x$, given by $e(x)y = x \wedge y$ and $c(x) : \wedge V' \to \wedge V'$, the *interior product* by $x$ (or the *contraction* with $x$), is the map transposed with respect to $e(\beta(x))$, i.e. $\langle y, c(x)z' \rangle = \langle \beta(x) \wedge y, z' \rangle$, for every $y \in \wedge V$ and $z' \in \wedge V'$. In particular,

$${}^t e(v) = c(v) \quad \text{for every} \quad v \in V. \tag{3}$$

The easy-to-check formulae

$$\beta \circ e(v) = e(v) \circ \alpha \circ \beta \quad \text{and} \quad \beta \circ c(v) = -c(v) \circ \alpha \circ \beta$$

are useful in computations. The maps $e(z')$ and $c(z')$ are similarly defined. If $v' \in V'$, then $c(v')$ is an anti-derivation, of degree $-1$, of the $\mathbb{Z}$-graded algebra $\wedge V$ and

$$e(v) \circ c(v') + c(v') \circ e(v) = \langle v, v' \rangle \, \mathrm{id}_{\wedge V}$$

for every $v \in V$.

*1.1.3. Hodge duality*

If the vector space $V$ is $n$-dimensional, then a *volume element* $\varepsilon$ on $V$ is a non-zero $n$-form, $\varepsilon \in \wedge^n V'$. We define the (modified[3]) *Hodge isomorphism*

[3] The modification, which consists in using, in formula (5), the expression $\beta(x) \wedge y$ instead of $x \wedge y$, is in agreement with the Kähler definition of the Hodge dual in terms of Clifford

relative to $\varepsilon$,

$$\varepsilon : \wedge V \to \wedge V', \quad \text{by} \quad \varepsilon(x) = c(x)\varepsilon, \tag{4}$$

i.e.

$$\langle \beta(x) \wedge y, \varepsilon \rangle = \langle y, \varepsilon(x) \rangle \quad \text{for every} \quad x, y \in \wedge V. \tag{5}$$

The abuse of notation implied by (4) is justified by $\varepsilon(1) = \varepsilon$. From $\beta(\varepsilon) = (-1)^{n(n-1)/2}\varepsilon$ and $\langle \beta(x), \beta(x') \rangle = \langle x, x' \rangle$ one obtains

$$^t\varepsilon = (-1)^{n(n-1)/2}\varepsilon. \tag{6}$$

Moreover, the definitions imply

$$\varepsilon \circ e(v) = c(v) \circ \varepsilon \quad \text{and} \quad \varepsilon \circ c(v') = e(v') \circ \varepsilon \tag{7}$$

for every $v \in V$ and $v' \in V'$. If $(u_1, \dots, u_k)$ is a linear basis in a vector subspace $U$ of $V$, then $\varepsilon(u_1 \wedge \cdots \wedge u_k)$ is of the form $u'_{k+1} \wedge \cdots \wedge u'_n$, where $u'_i \in V'$, $i = k+1, \dots, n$ and $U^\circ = \operatorname{span}\{u'_{k+1}, \dots, u'_n\}$.

If $n = 2p$ , then there is the bilinear map

$$\wedge^p V \times \wedge^p V \to K, \quad (x, y) \mapsto \langle x, \varepsilon(y) \rangle, \tag{8}$$

which is symmetric or skew, depending on whether $p$ is even or odd.

*1.1.4. Low-dimensional examples*

We denote by End $(S)$ the algebra of all $K$-linear endomorphisms of a finite-dimensional vector space $S$. If $x \in S$ and $x' \in S'$, then there is the endomorphism $x \otimes x'$ such that

$$(x \otimes x')(y) = \langle y, x' \rangle x \quad \text{for every} \quad y \in S.$$

Clearly, $\operatorname{Tr}(x \otimes x') = \langle x, x' \rangle$, $^t(x \otimes x') = x' \otimes x$ and

$$u \circ (x \otimes x') \circ v = u(x) \otimes {}^t v(x') \tag{9}$$

for every $x \in S$, $x' \in S'$, and $u, v \in \operatorname{End}(S)$.

Consider now a *two*-dimensional vector space $S$ over $K$, with a volume element $\varepsilon$ and the associated *four*-dimensional vector space $W = \operatorname{End}(S)$. The quadratic form

$$h = \det : \operatorname{End}(S) \to K$$

defined by

$$w(x) \wedge w(y) = (\det w)\, x \wedge y, \quad \text{where} \quad x, y \in S \quad \text{and} \quad w \in W, \tag{10}$$

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multiplication by the volume element [BT1,RoT2]; see also (23). It leads to the simple formula (6) for the transpose of $\varepsilon$.

is non-singular; from (5) and (10), by evaluating ${}^t w \circ \varepsilon \circ w$ on $x$ and $y$, one obtains $\langle y, {}^t w \circ \varepsilon \circ w(x) \rangle = \langle w(x) \wedge w(y), \varepsilon \rangle = \langle y, \varepsilon(x) \rangle \det w$, or

$$w' \circ w = (\det w)\, \mathrm{id}_S, \tag{11}$$

where $w' = \varepsilon^{-1} \circ {}^t w \circ \varepsilon | S$ is also an element of $W$. The four-dimensional vector space of *Dirac spinors*, associated with $W$, is the direct sum, $R = S \oplus S$, of two copies of spaces of *Weyl spinors*. Let

$$\gamma(w) : R \to R \quad \text{be defined by} \quad \gamma(w) = \begin{pmatrix} 0 & w \\ w' & 0 \end{pmatrix},$$

then (11) gives

$$\gamma(w)^2 = h(w)\, \mathrm{id}_R. \tag{12}$$

The endomorphisms $\gamma(w)$, where $w \in W$, generate the algebra End $(S \oplus S)$, which, in this context, is the *Clifford algebra* of (End $(S)$, det) over $K$. From now on, to the end of this paragraph, we write $z = (x, y) \in S \oplus S$ and identify $x$ with $(x, 0)$ and $y$ with $(0, y)$. By virtue of (12), the set

$$N(z) = \{ w \in W : \gamma(w) z = 0 \}, \quad \text{where} \quad z \in R^\times,$$

is a totally isotropic subspace of $W$. Moreover, $N(x, y) = N(x) \cap N(y)$. If $x \neq 0$, then $N(x)$ is maximal among totally isotropic subspaces of $W$: it is, indeed, two-dimensional because it can be identified with $x \otimes S'$. Similarly, if $y \neq 0$, then $N(y) = S \otimes \varepsilon(y)$. Therefore, if both $x$ and $y$ are non-zero, then $N(x) \cap N(y) = K x \otimes \varepsilon(y)$ is one-dimensional.

Consider next a *four*-dimensional space $S$ over $K$, with a volume element $\varepsilon \in \wedge^4 S'$. According to (8), the *six*-dimensional vector space $W = \wedge^2 S$ has a quadratic form defined by the Pfaffian, $h = \mathrm{Pf}$,

$$\mathrm{Pf}(w) = \tfrac{1}{2} \langle w \wedge w, \varepsilon \rangle, \quad \text{where} \quad w \in W.$$

Since now $W \subset \mathrm{Hom}(S', S)$ and $W' \subset \mathrm{Hom}(S, S')$, there is a composition map, $W \times W' \to \mathrm{End}(S)$, such that, for every $x_1, x_2 \in S$ and $x_1', x_2' \in S'$, there holds

$$\begin{aligned}(x_1 \wedge x_2) \circ (x_1' \wedge x_2') \\ = x_1 \otimes x_1' \langle x_2, x_2' \rangle + x_2 \otimes x_2' \langle x_1, x_1' \rangle - x_1 \otimes x_2' \langle x_2, x_1' \rangle - x_2 \otimes x_1' \langle x_1, x_2' \rangle.\end{aligned}$$

There is a similar composition map with $W, W'$ and $S$ replaced by $W', W$ and $S'$, respectively. For every $w \in W$ one has

$$w \circ \varepsilon(w) = h(w)\, \mathrm{id}_S \quad \text{and} \quad \varepsilon(w) \circ w = h(w)\, \mathrm{id}_{S'}.$$

Defining now the space of Dirac spinors as $R = S \oplus S'$ and

$$\gamma(w) = \begin{pmatrix} 0 & w \\ \varepsilon(w) & 0 \end{pmatrix}.$$

one sees that Eq. (12) holds again and End $(S \oplus S')$ is the Clifford algebra of $(\wedge^2 S, \mathrm{Pf})$. Putting $z = (x, x') \in S \oplus S'$ and making similar identifications to those of the preceding paragraph, one obtains that, for $x \in S^\times$,

$$N(x) = \{w \in W : w \wedge x = 0\}$$

is a three-dimensional totally isotropic space, consisting of all bivectors of the form $x \wedge y$, where $y \in S$. Similarly, if $x' \in S'^\times$, then

$$N(x') = \{w \in W : w(x') = 0\}$$

is three-dimensional totally isotropic, and can be identified with $\wedge^2 x'^\circ$. Therefore, if both $x$ and $x'$ are $\neq 0$, then

$$N(x) \cap N(x') = \begin{cases} \{0\} & \text{if} \quad \langle x, x'\rangle \neq 0 \\ Kx \wedge x'^\circ & \text{is 2-dimensional if} \quad \langle x, x'\rangle = 0. \end{cases}$$

The above construction, used in twistor theory [PeR], prolongs to a seven-dimensional vector space $U = W \oplus Ke_7$, where $W = \wedge^2 S$ and the quadratic form on $U$ extends the Pfaffian and makes the unit vector $e_7$ orthogonal to $W$. One represents $e_7$ in $R = S \oplus S'$ by the endomorphism

$$\gamma(e_7) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $z = (x, x') \neq 0$ and $u = w + \lambda e_7$, then

$$N(z) = \{u \in U : w(x') + \lambda x = 0 \quad \text{and} \quad \varepsilon(w)(x) - \lambda x' = 0\}$$

is totally isotropic and maximal (i.e. three-dimensional) if, and only if, $\langle x, x'\rangle = 0$; otherwise, $N(z)$ reduces to the zero vector. Seven is the least dimension of a vector space, with a quadratic form of maximal index, which admits spinors of the latter type: for $m = 3$ and 5 every spinor and for $m = 2, 4$ and 6 every Weyl spinor $x$ is pure: its nullity—the dimension of $N(x)$—is equal to the index of the quadratic form.

### *1.2. Historical remarks*

There is a 'prehistory' of spinors: the germs of the ideas of spin groups and their representations can be found in the work of L. Euler, O. Rodrigues, W.R. Hamilton, A. Cayley, W.K. Clifford and R.O. Lipschitz, see [BT1,T] for references and further remarks on this subject. Élie Cartan [C1] discovered what are now called spinor representations of the complex Lie algebras so$(n)$, $n > 2$. Spinors owe their name and fame to physicists. According to B.L. van der Waerden [W3], the name *spinor* is due to P. Ehrenfest, who suggested, on a visit to Göttingen, to develop a spinor analysis analogous to tensor calculus [W1]. During the first 15 years that followed the discovery of the spin of the electron, important work on spinors was done by Pauli, Dirac, Weyl, Fock, Bargmann, Schrödinger, Majorana, Laporte and Uhlenbeck, Infeld and van der Waerden, Haantjes and Schouten, and several other authors; a good source of references to that period is [Co]. The connections between spinors, totally isotropic spaces and projective geometry seem to have been clearly stated, for the first time, by O. Veblen [V1,V2] and developed in seminar lectures at Princeton given jointly with J.W. Givens [VG]. The latter prepared a Ph.D. thesis [G], which, in a section on the *Geometry of a generalization of the Plücker–Klein correspondence*, contains remarks that may have influenced É. Cartan in his work on pure[4] spinors; by some accident, Ref. [G] appears in the French original [C3], but not in the English translation [C4] of *Leçons sur la théorie des spineurs*. At about the same time, Brauer and Weyl [BrWe] gave a description of the representations of the groups $\mathrm{Spin}_m$; they made clear the role of the Clifford algebras in their construction and found the decompositions of the tensor products of the representations into irreducible parts.

Cartan's Lectures [C3,C4] contain an exposition of the notion of a pure spinor and are rich in geometrical ideas; some of the proofs there are outlined only and the underlying field is restricted to be either $\mathbb{C}$ or $\mathbb{R}$. These shortcomings have been overcome by C. Chevalley, who based his *Algebraic theory of spinors* [Ch] on the notion of minimal, one-sided ideals of Clifford algebras, an idea considered earlier by M. Riesz [Ri] in the context of the Dirac equation in the theory of general relativity and, less explicitly, by several physicists; see [S] and the references given there. Very early, spinor fields were introduced, in a 'local' manner, on Lorentzian manifolds of Einstein's theory. For a considerable length of time, the lack of a global definition, needed in the context of manifolds with non-trivial topology, and the subtle differences between tensors and spinors, baffled mathematicians and physicists alike; compare, e.g., an opinion expressed in 1928 by C.G. Darwin (quoted in [BT1, p. 4]), the footnotes at the end of [C3], the discussions on the Lie derivatives of spinors or a paper that appeared in an early volume of this Journal [Mo]. The proper definition, intimated by Cartan, has been given, in the 1950s, in terms of fibre bundles; see [LM] for a presentation of the notion of a spin structure and of the applications to geometry of global properties of the Dirac operator. This book is also a good guide to the 'modern' period of the work on Clifford

[4] In fact, Cartan used the expression *spineur simple*; the name 'pure spinor' is due to Chevalley and seems to have been generally accepted even though it is somewhat disturbing to think of Dirac spinors as being 'impure', cf. [BT2].

algebras and their representations, on the index theorem in the context of spin structures, on spin cobordism and on harmonic spinors.

The projective-geometrical and 'optical' aspects of spinors, introduced by Veblen and Cartan, have led to important applications in the theory of general relativity and Yang–Mills theory, mainly through the work of Roger Penrose and his school, and the development of his twistor theory [A,PeR].

### *1.3. A short remark on applications of pure spinors in physics*

In this paper, we consider the problem of classifying Weyl spinors according to their nullity. The main results are summarized in the theorems in Section 3. Most of the time, we restrict ourselves here to the field of complex numbers, but important applications of pure spinors are associated with real structures [BeTu,KoT,PeR] . Put very briefly, they rely on the following [NuT,T]: If $W$ is the complexification of a real space $V$ with a scalar product of signature $(2p+\delta, 2q+\delta)$, where $\delta = 0$ or $1$ and $p+q+\delta = n$, then the *real index* of a pure spinor $x \neq 0$, $r = \dim(N(x) \cap \overline{N(x)})$, in the generic case equals $\delta$. Therefore, the direction of a pure spinor in a general position defines in $V$ a complex ($\delta = 0$) or an optical ($\delta = 1$) structure. These observations can be applied to a smooth, orientable $2n$-dimensional spin manifold $\mathcal{M}$ with a bundle of directions of generic pure spinors. A section of this bundle—if it exists—defines an almost complex or an almost optical geometry, depending on whether $r = 0$ or $1$. With such a section one associates a bundle $\mathcal{N}$ of maximal, totally isotropic subspaces of the complexified tangent spaces to $\mathcal{M}$. Denoting by $\mathcal{Z}$ the module of sections of the bundle $\mathcal{N}$, one considers the integrability conditions $[\mathcal{Z}, \mathcal{Z}] \subset \mathcal{Z}$. In the pseudo-Euclidean case ($\delta = 0$), the condition is equivalent to the vanishing of the Nijenhuis tensor of the almost complex structure; in the Lorentzian, four-dimensional case, it is related to the geodetic, shear-free properties of the trajectories of the real line bundle $\mathrm{Re}\,(\mathcal{N} \cap \overline{\mathcal{N}}) \to \mathcal{M}$. In the theory of general relativity, congruences of shear-free isotropic geodesics play an important role in the study of algebraically special gravitational fields; see [PeR,RoT1] and the references given there.

## 2. Grassmannians, Clifford algebras and groups

With a vector space $W$ over $K$ one associates the $k$th *Grassmannian* $\mathrm{Gr}^k(W)$ of all $k$-dimensional vector subspaces ($k$-planes) of $W$ and the *total Grassmannian* $\mathrm{Gr}(W) = \bigcup_k \mathrm{Gr}^k(W)$. The general linear group $\mathrm{GL}(W)$ acts transitively on each $\mathrm{Gr}^k(W)$. In particular, $\mathrm{Gr}^1(W) = \mathrm{P}(W)$ is the *projective space* associated with $W$. There is a canonical map dir: $W^\times \to \mathrm{P}(W)$.

### 2.1. The Witt theorem and quadric Grassmannians

The *isotropic cone* in a quadratic space $(W, h)$ is the set $W_{\text{cone}} = \{w \in W : h(w) = 0\}$ and the $k$th *quadric Grassmannian* $\mathrm{Q}^k(W, h) \subset \mathrm{Gr}^k(W)$ is defined as the set of all totally isotropic $k$-planes in $W$ [Pr]. In particular, $\mathrm{Q}^1(W, h) \subset \mathrm{P}(W)$ is the *quadric* and the map dir restricts to $W_{\text{cone}} \setminus \{0\} \to \mathrm{Q}^1(W, h)$. The *total quadric Grassmannian* is $\mathrm{Q}(W, h) = \bigcup_k \mathrm{Q}^k(W, h)$.

**Proposition 1** (Witt). *Let $W$ be a $2n$-dimensional vector space over $K$ with a neutral quadratic form $h$. Then*

*(i) the group $\mathrm{O}(W, h)$ of orthogonal automorphisms of $(W, h)$ acts transitively on each quadric Grassmannian $\mathrm{Q}^k(W, h)$, $k = 1, \ldots, n$;*

*(ii) if $T$ is a maximal totally isotropic (mti) subspace of $W$, with a linear basis $(t_1, \ldots, t_n)$, then there exists another mti subspace $U$ of $W$, with a linear basis $(u_1, \ldots, u_n)$ such that $W = T \oplus U$ and $g(t_i, u_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$;*

*(iii) if $T_1$ is a subspace of $T$, then there is a subspace $U_1$ of $U$ such that $h|T_1 \oplus U_1$ is non-degenerate.*

*Proof.* Can be found in [Bour, §4, no. 2 and 3].

**Lemma 1.** *(i) Every totally isotropic subspace $V$ in a vector space $W$ with a neutral quadratic form can be represented as an intersection of two mti subspaces of $W$.*

*(ii) If $T_1$ and $T_2$ are two totally isotropic subspaces of $W$ and $T_2 \not\subset T_1$, then there exists an mti subspace $U \subset W$ such that $T_1 \subset U$, but $T_2 \not\subset U$.*

*Proof.* The first statement can be proved directly by using the Witt decomposition and basis. Assuming that $W$ is $2n$-dimensional and using the notation of part (ii) of Proposition 1, one can represent a $k$-dimensional totally isotropic space as

$$\operatorname{span}\{t_1, \ldots, t_k\} = \operatorname{span}\{t_1, \ldots, t_n\} \cap \operatorname{span}\{t_1, \ldots, t_k, u_{k+1}, \ldots, u_n\}.$$

The second follows from the first: let $T_i = V_i \cap U_i$, where $V_i$ and $U_i$, $i = 1, 2$, are *mti* spaces. Since $T_2 \not\subset T_1$, at least one of the following is true: $T_2 \not\subset V_1$ or $T_2 \not\subset U_1$. □

### 2.2. Clifford algebras

The following two statements are classical; see, e.g., [Bour].

**Proposition 2.** *Let $\mathcal{T}(W)$ be the tensor algebra of a vector space $W$ over $K$ and let $\mathcal{I}(h)$ be the bilateral ideal of $\mathcal{T}(W)$ generated by all elements of the form*

*$w \otimes w - h(w).1$, where $h$ is a quadratic form on $W$. The Clifford algebra of the quadratic space $(W, h)$,*

$$\mathrm{Cliff}(W, h) = \mathcal{T}(W)/\mathcal{I}(h)$$

*is an algebra over $K$, containing $W$ as a vector subspace, and having the universal property: if $\mathcal{A}$ is an algebra over $K$ and $f : W \to \mathcal{A}$ is a Clifford map, i.e. a linear map such that $f(w)^2 = h(w).1$ for every $w \in W$, then there is a homomorphism $\hat{f} : \mathrm{Cliff}(W, h) \to \mathcal{A}$ of algebras extending $f$, i.e. such that $\hat{f}|W = f$.*

The Clifford map $w \mapsto -w$ extends to the *$\mathbb{Z}_2$-grading automorphism* $\alpha_h$ of $\mathrm{Cliff}(W, h)$; the canonical injection of $W$ into the algebra opposite to $\mathrm{Cliff}(W, h)$ gives rise, in a similar manner, to the *main antiautomorphism* $\beta_h$. The even subalgebra is

$$\mathrm{Cliff}^+(W, h) = \{a \in \mathrm{Cliff}(W, h) : \alpha_h(a) = a\}.$$

**Proposition 3.** *There is an isomorphism of vector spaces*

$$\imath : \mathrm{Cliff}(W, h) \to \wedge W \tag{13}$$

*obtained by extending the Clifford map*

$$f : W \to \mathrm{End} \wedge W, \quad f(w) = e(w) + c(g(w)),$$

*to the homomorphism $\hat{f} : \mathrm{Cliff}(W, h) \to \mathrm{End} \wedge W$ and evaluating it on the unit element 1 of $\wedge W$, $\imath(a) = \hat{f}(a)1$. Moreover $\imath$ is the identity map on $K \dotplus W$,*

$$\imath(wa) = e(w)\imath(a) + c(g(w))\imath(a), \tag{14}$$

*for every $w \in W$ and $a \in \mathrm{Cliff}(W, h)$,*

$$\imath(uv - vu) = 2\imath(u) \wedge \imath(v) \quad \text{for} \quad u, v \in W$$

*and*

$$\imath \circ \alpha_h = \alpha \circ \imath, \quad \imath \circ \beta_h = \beta \circ \imath.$$

## *2.3. The Clifford group*

Let $u \in W$ be a non-isotropic vector; the map

$$w \mapsto \rho(u)w = -uwu^{-1}, \tag{15}$$

of $W$ into itself, is a reflection in the hyperplane orthogonal to $u$. The multiplication on the right side of (15) is in $\mathrm{Cliff}(W, h)$ and $u^{-1} = h(u)^{-1}u$. The same map, expressed in terms appropriate to the Grassmann algebra, reads $w \mapsto \sigma(u)w$, where

$$\sigma(u) = h(u)^{-1}(e(u) + c(g(u)) \circ (e(u) - c(g(u))$$
$$= \mathrm{id}_W - 2h(u)^{-1}e(u) \circ c(g(u)). \tag{16}$$

The latter map extends to the automorphism $\wedge\sigma(u)$ of the Grassmann algebra.

The *Clifford group* $\mathrm{G}(W, h)$ is defined as the subset of $\mathrm{Cliff}(W, h)$ consisting of the products of elements of all finite sequences of non-isotropic vectors; multiplication in the group is induced by that in the algebra. If $a \in \mathrm{G}(W, h)$, then $\mu(a) = \beta_h(a)a \in K^\times$ is the *norm* of $a$. With $\rho$ defined by

$$\rho(a)w = \alpha(a)wa^{-1} \tag{17}$$

one has the exact sequence of group homomorphisms

$$1 \to K^\times \to \mathrm{G}(W, h) \xrightarrow{\rho} \mathrm{O}(W, h) \to 1.$$

For $a = u_1 \dots u_k \in \mathrm{G}(W, h)$ one puts $\sigma(a) = \sigma(u_1) \circ \dots \circ \sigma(u_k)$; this defines a representation of the Clifford group in $\wedge W$. The even Clifford group is

$$\mathrm{G}^+(W, h) = \mathrm{G}(W, h) \cap \mathrm{Cliff}^+(W, h).$$

and one has

$$\imath(aba^{-1}) = \wedge\sigma(a) \circ \imath(b) \tag{18}$$

for every $a \in \mathrm{G}^+(W, h)$ and $b \in \mathrm{Cliff}(W, h)$.

### *2.4. The hyperbolic model of neutral spaces*

It is convenient to have a 'universal' model of vector spaces with a neutral quadratic form; such a 'hyperbolic' model and the corresponding Clifford algebra, are described in the following Proposition.

**Proposition 4.** *Let $V$ be an $n$-dimensional vector space and let $W = V \oplus V'$ be given the canonical, neutral quadratic form $h$,*

$$h(v + v') = \langle v, v' \rangle$$

*for every $v \in V$ and $v' \in V'$. There is an isomorphism of algebras*

$$\gamma : \mathrm{Cliff}(W, h) \to \mathrm{End} \wedge V \tag{19}$$

*obtained by extending the Clifford map $V \oplus V' \to \mathrm{End} \wedge V$ such that $v + v' \mapsto e(v) + c(v')$. For every $a \in \mathrm{Cliff}(W, h)$ one has*

$$^t\gamma(\beta_h(a)) = \varepsilon \circ \gamma(a) \circ \varepsilon^{-1}. \tag{20}$$

*Proof.* This is also well-known (see, e.g., [Ba]): both $V$ and $V'$ are *mti* spaces and (20) is a consequence of (3) and (7). One says that (19) is a representation of the Clifford algebra of a neutral quadratic space in the space $R = \wedge V$ of *Dirac spinors*. This representation is faithful and irreducible.

The dual $W'$ of $W = V \oplus V'$ can be identified with the space $W$ itself, the pairing being defined by

$$\langle v_1 + v_1', v_2 + v_2' \rangle = \tfrac{1}{2}\langle v_1, v_2' \rangle + \tfrac{1}{2}\langle v_2, v_1' \rangle \quad \text{for} \quad v_i \in V,\ v_i' \in V',\ i = 1, 2.$$

The isomorphism $g$, associated with $h$ by (1), reduces now to the identity. Let

$$(v_i),\ i = 1, \ldots, n, \text{ be a linear basis in } V \\ \text{and } (v_i') \text{ the associated dual basis in } V'. \tag{21}$$

The volume element

$$\eta = (v_1'v_1 - v_1v_1') \cdots (v_n'v_n - v_nv_n')$$

satisfies

$$\eta^2 = 1 \quad \text{and} \quad \imath(\eta) = 2^n v_1' \wedge v_1 \wedge \cdots \wedge v_n' \wedge v_n.$$

Moreover, since $\gamma(v_i'v_i - v_iv_i')v_j = v_j$ for $i \neq j$ and $-v_j$ for $i = j$, one sees that, defining the *helicity automorphism* by $\Gamma = \gamma(\eta)$, one has $\Gamma(x) = (-1)^{\chi(x)}x$, where $\chi(x) = 0$ or $1$ for $x \in \wedge^+ V$ or $\wedge^- V$, respectively. In this context, one says that (2) is the decomposition of the space of Dirac spinors into two spaces of *Weyl spinors* of positive and negative helicities. Denoting by $S$ the space of Weyl spinors of positive helicity, one obtains, by restriction of (19), the representation of the even subalgebra,

$$\gamma : \mathrm{Cliff}^+(W, h) \to \mathrm{End}(S), \quad \text{where} \quad S = \wedge^+ V. \tag{22}$$

The *Kähler dual* of an element of $\wedge W$ is defined by

$$\star\imath(a) = \imath(\eta a) \quad \text{for} \quad a \in \mathrm{Cliff}(W, h) \tag{23}$$

so that

$$\star\star = \mathrm{id} \quad \text{and} \quad \beta \circ \star = (-1)^n \star \circ \beta \circ \alpha. \tag{24}$$

## *2.5. The bilinear equivariant map*

Let $(W, h)$ be the quadratic space described in Prop. 4. The representation (19) and the isomorphism (13) define an isomorphism of vector spaces

$$\kappa = \imath \circ \gamma^{-1} : \wedge V \otimes \wedge V' \to \wedge(V \oplus V'),$$

which is different from the canonical, linear isomorphism among these spaces obtained by extending the map $x \otimes y' \mapsto x \wedge y'$, where $x \in \wedge V$ and $y' \in \wedge V'$.

For example, if $(x_A)$, where $A = 1, \ldots, 2^n$, is a linear basis in $\wedge V$ and $(x'_A)$ is the corresponding dual basis in $\wedge V'$, then $\kappa(\sum_A x_A \otimes x'_A) = 1_{\wedge W}$. With a pair $(x, y)$ of spinors we associate the multivector

$$E(x, y) = \kappa(x \otimes \varepsilon(y)), \quad x, y \in \wedge V, \tag{25}$$

and denote by $E_k(x, y)$ the component of $E(x, y)$ in $\wedge^k W$. We define the 'quadratic covariant' associated with a spinor $x \in \wedge V$ by $F(x) = E(x, x)$ and put $F_k(x) = E_k(x, x)$.

**Theorem 1.** *The bilinear map* $E : \wedge V \times \wedge V \to \wedge W$ *defined by (25) has the following properties holding for every* $x, y \in \wedge V$, $a \in \mathrm{G}^+(W, h)$ *and* $w \in W$:

*(i) it is equivariant with respect to the action of* $\mathrm{G}^+(W, h)$,

$$E(\gamma(a)x, \gamma(a)y) = \mu(a).\wedge\sigma(a) \circ E(x, y);$$

*(ii)* $E(\gamma(w)x, y) = (e(w) + c(g(w))) \circ E(x, y)$;
*(iii)* $E(y, x) = (-1)^{n(n-1)/2}\beta \circ E(x, y)$;
*(iv)* $E(\Gamma x, y) = \star E(x, y)$;
*(v)* $E(x, \Gamma y) = (-1)^n \alpha \circ E(\Gamma x, y)$;
*(vi) if x and y are Weyl spinors, then*

$$\chi(x) + \chi(y) + k - n \equiv 1 \bmod 2 \ \textit{implies}\ E_k(x, y) = 0;$$

*(vii) if x is a Weyl spinor, then* $F_k(x) = 0$ *unless* $k \equiv n \bmod 4$.

*Proof.*

(i) From the definition of $E$ and (9), one has

$$E(\gamma(a)x, \gamma(a)y) = \imath(a\gamma^{-1}(x \otimes \varepsilon(y))\beta_h(a))$$

and the result follows from (18).

(ii) This is a consequence of (9) and (14).

(iii) Use $y \otimes \varepsilon(x) = \varepsilon^{-1} \circ {}^t(x \otimes \varepsilon(y)) \circ {}^t\varepsilon$, (6) and (20).

(iv) Follows at once from (23).

(v) Follows from (iii), (iv) and (24).

(vi) Assuming that $x$ and $y$ are Weyl, from (iv) and (v) one obtains

$$E(x, y) = E(x, \Gamma^2 y) = (-1)^n \alpha \circ E(\Gamma x; \Gamma y)$$
$$= (-1)^{\chi(x) + \chi(y) + n} \alpha \circ E(x, y).$$

(vii) Note that (iii) can be written as

$$E_k(y, x) = (-1)^{(n-k)(n+k-1)/2} E_k(x, y)$$

and $\frac{1}{2}(n-k)(n+k-1) \equiv 1 \bmod 2$ for $k-n \equiv 2 \bmod 4$; if $k-n \equiv 1 \bmod 2$, then $F_k(x)$ vanishes by virtue of (vi). □

Note that if $x$ is a Weyl spinor of nullity $\nu$, $N(x) = \mathrm{span}\{w_1, \dots, w_\nu\}$, say, then there exists $\Phi \in \wedge W$ such that

$$F(x) = w_1 \wedge \cdots \wedge w_\nu \wedge \Phi.$$

*2.6. The Cartan–Chevalley theory of pure spinors*

From now on, through the end of the paper, we restrict ourselves to **complex**[5] vector spaces of **even dimension** $m = 2n > 0$ and consider Clifford algebras, spin groups and spinors associated with such spaces. The ground field being fixed, we use a notation emphasizing only the dimension of the underlying space. Thus the Clifford algebra of the quadratic space $(\mathbb{C}^m, h)$, with $h$ non-degenerate, is denoted by $\mathrm{Cliff}_m$ and $\mathrm{G}_m$ is the corresponding Clifford group, $\mathrm{GL}_m$ is the general linear group, etc. The Pin and Spin groups are defined by

$$\mathrm{Pin}_m = \{a \in \mathrm{G}_m : \mu(a) = 1\}$$

and

$$\mathrm{Spin}_m = \mathrm{Pin}_m \cap \mathrm{Cliff}_m^+,$$

respectively. We write $\mathrm{Q}_n^k$ instead of $\mathrm{Q}^k(\mathbb{C}^{2n}, h)$.

Élie Cartan's theory of pure spinors can be summarized as follows. Let $(t_1, \dots, t_n)$ be a linear basis in an *mti* subspace $T$ of $W = V \dotplus V'$. Since the representation (19) is faithful, there exists a spinor $z \in R = \wedge V$ such that $x = \gamma(t_1 \cdots t_n) z \neq 0$. The spinor $x$ is pure, $N(x) = T$. For example, $N(1) = V'$. If $y$ is another spinor such that $N(y) = T$, then there is $\lambda \in \mathbb{C}^\times$ such that $y = \lambda x$. Therefore, there is a bijective correspondence between the set of directions of pure spinors and the quadric Grassmannian $\mathrm{Q}_n^n$, a complex manifold of dimension $\frac{1}{2}n(n-1)$. For every $x \in R$ and $a \in \mathrm{Pin}_m$ one has $N(\gamma(a)x) = \rho(a)N(x)$. Every pure spinor $x$ is a Weyl spinor, $\Gamma(x) = (-1)^{\chi(x)}x$; if $N(x) = \mathrm{span}\{t_1, \dots, t_n\}$, then $\star(t_1 \wedge \cdots \wedge t_n) = (-1)^{\chi(x)} t_1 \wedge \cdots \wedge t_n$. There is a bijective correspondence between the set of directions of pure spinors of positive helicity and the manifold $\mathrm{Q}_n^{n+}$ of self-dual (one also says: of positive helicity) *mti* subspaces in $W$. This manifold is one of two connected components of $\mathrm{Q}_n^n$. The groups $\mathrm{O}_{2n}$ and $\mathrm{SO}_{2n}$ act transitively on the spaces of directions of all pure spinors and pure spinors of a given helicity, respectively. Less obvious are the following facts.

**Proposition 5** (Cartan–Chevalley). *(i) A Weyl spinor $x \neq 0$ associated with $W = \mathbb{C}^{2n}$ is pure if, and only if, $F_k(x) = 0$ for $k \neq n$;*

[5] This assumption is not essential: all the following considerations can be formulated so as to be valid over a field of characteristic $\neq 2$. We prefer, however, to confine ourselves to complex geometry and use the concept of a manifold, more familiar to physicists than the algebraic geometers' notion of *varieties*.

*(ii) if $x$ and $y$ are pure spinors, then the dimension of $N(x) \cap N(y)$ is the least integer $k$ such that $E_k(x,y) \neq 0$; moreover $E_k(x,y) = v_1 \wedge \cdots \wedge v_k$, where the vectors $v_1, \ldots, v_k$ are such that $N(x) \cap N(y) = \mathrm{span}\{v_1, \ldots, v_k\}$ and one has $\chi(x) + \chi(y) + k - n \equiv 0 \bmod 2$;*

*(iii) if $x$ and $y$ are linearly independent pure spinors, then $x + y$ is pure if, and only if, $\dim(N(x) \cap N(y)) = n - 2$; if this condition is satisfied, then $N(x+y) \cap N(y) = N(x) \cap N(y)$.*

*Proof.* One finds it in [C3,Ch,BeTu].

### 2.7. Orbits of the Spin groups in low dimensions

A considerable amount of work has been done on the classification of the orbits of the Spin groups associated with low-dimensional spaces; essentially everything is known up to dimension 14 [Ig,Pp]. For our purposes it is enough to summarize the results for even dimensions $\leq 12$.

By restricting (22) to $\mathrm{Spin}_{2n} \subset \mathrm{Cliff}_{2n}^{+}$, one obtains the Weyl representation of the group in the $2^{n-1}$-dimensional space $S$ of spinors of positive helicity,

$$\gamma : \mathrm{Spin}_{2n} \to \mathrm{GL}(S), \quad S = \wedge^{+} V, \quad V = \mathbb{C}^n. \tag{26}$$

**Proposition 6.** *Consider the action of the group $\mathrm{Spin}_{2n}$ in the space $S^\times$ of non-zero Weyl spinors, defined by the representation (26). Then*

*(i) For $n = 1, 2$ and 3, the action is transitive.*

*(ii) If $n = 4$, then, for every $\lambda \in \mathbb{C}$, there is a 7-dimensional orbit $\{x \in S^\times : F_0(x) = \lambda\}$ and $\dim N(x) = 4$ or 0 depending on whether $\lambda = 0$ or $\lambda \neq 0$.*

*(iii) For $n = 5$ there are two orbits: that of pure spinors, characterized by $F_1(x) = 0$ and the orbit of spinors of nullity 1; if $F_1(x) \neq 0$, then $N(x) = \mathbb{C} F_1(x)$.*

*(iv) For $n = 6$ there is the Igusa invariant $J(x)$ defined by*

$$\star J(x) = \star F_2(x) \wedge F_2(x).$$

*For every $\lambda \in \mathbb{C}^\times$ there is one orbit $\{x \in S : J(x) = \lambda\}$ of dimension 31. Besides those, there are three orbits on which the invariant vanishes:*

*(a) the 16-dimensional orbit of pure spinors, characterized by $F_2(x) = 0$;*

*(b) the 25-dimensional orbit of spinors of nullity 2, characterized by $F_2(x) \neq 0$ and $F_2(x) \wedge F_2(x) = 0$;*

*(c) a 31-dimensional orbit of spinors of zero nullity, characterized by $F_2(x) \wedge F_2(x) \neq 0$.*

*Proof.* (i) This is well-known: the groups $\mathrm{Spin}_2 = \mathrm{GL}_1$, $\mathrm{Spin}_4 = \mathrm{SL}_2 \times \mathrm{SL}_2$ and $\mathrm{Spin}_6 = \mathrm{SL}_4$ act transitively on their respective spaces of non-zero Weyl spinors. (ii) This is a well-known manifestation of triality. (iv) and (v) follow, respectively, from Propositions 2 and 3 in [Ig].

## 3. Partially pure spinors

We continue using the notation introduced in §2.4 and §2.6. In particular, $R = \wedge V$ and $S = \wedge^+ V$ are the spaces of Dirac and of Weyl spinors of positive helicity, respectively, associated with the $2n$-dimensional vector space $W = V \oplus V'$.

Lemma 1 can be completed by the following

**Lemma 2.** *(i) If $T \in \mathrm{Q}_n^{n-1}$, then there exist exactly two mti subspaces $T_1$ and $T_2$ containing $T$; they are of opposite helicity.*

*(ii) If $T \in \mathrm{Q}_n^k$, where $k \leq n-2$, then $T$ can be represented as an intersection of either two or three mti subspaces of positive helicity, depending on whether $n-k$ is even or odd.*

*Proof.* (i) Decompose $W$ into a direct sum $W_1 \oplus W_2$ of orthogonal subspaces such that $h|W_i$ $(i = 1, 2)$ is non-degenerate and $T \subset W_1$. Then $\dim W_2 = 2$ and $W_{2\,\mathrm{cone}} = L_1 \cup L_2$ with $\dim L_1 = \dim L_2 = 1$. The subspaces $T_i = W_1 \oplus L_i$ $(i = 1, 2)$ are *mti* and of opposite helicities because one of them can be transformed onto the other by an isometry of $W$ which reduces to the identity on $W_1$ and is a reflection on $W_2$, interchanging $L_1$ and $L_2$.

(ii) This can be proved along lines similar to the proof of (i), or, by adapting a basis (21) to $T$ and giving an explicit construction of the intersecting *mti*s. For example, if $n$ is even and $k$ is odd, then

$$\begin{aligned}\mathrm{span}\{v_1, \ldots, v_k\} = \mathrm{span}\{v_1, \ldots, v_n\} &\cap \mathrm{span}\{v_1, \ldots, v_k, v_{k+1}, v'_{k+2}, \ldots, v'_n\} \\ &\cap \mathrm{span}\{v_1, \ldots, v_k, v_{k+2}, v'_{k+1}, v'_{k+3}, \ldots, v'_n\}.\end{aligned}$$

**Theorem 2.** *Consider the maps:*

$$\psi : \mathrm{Q}(W, h) \to \mathrm{Gr}(R) \quad \textit{and} \quad \varphi : \mathrm{Q}(W, h) \to \mathrm{Gr}(S)$$

*defined, respectively, by*

$$\psi(T) = \{x \in R : \gamma(t)x = 0 \ \textit{for every}\ t \in T\} \quad \textit{and} \quad \varphi(T) = \psi(T) \cap S.$$

*(i) The map $\psi$ is injective.*

*(ii) The map $\varphi$ restricted to $\bigcup_{k=1}^{n-2} \mathrm{Q}^k(W, h)$ is injective.*

*(iii) If $T \in \mathrm{Q}_n^{n-k}$, where $k = 1, \ldots, n$, then $\dim \varphi(T) = 2^{k-1}$. Moreover, if $T \subset V'$, then the dual $U'$ of $U = T^\perp \cap V$ can be identified with a subspace of $V'$ complementary to $T$. The restriction of $h$ to $U \oplus U' \subset W$ is non-degenerate. By restriction, (26) gives a representation of the group $\mathrm{Spin}_{2k} \subset \mathrm{Spin}_{2n}$ in a space of Weyl spinors $\wedge^+ U = \varphi(T)$. Defining*

$$N_U(x) = \{w \in U \oplus U' : \gamma(w)x = 0\},\ \nu_U = \dim N_U(x), \quad \textit{for}\ x \in (\wedge^+ U)^\times,$$

*one has*

$$n - \nu = k - \nu_U. \tag{27}$$

*Proof.* (i) Let $T_1, T_2$ and $U$ be as in part (ii) of Lemma 1. A pure spinor $x \in \psi(U)$ belongs to $\psi(T_1)$, but not to $\psi(T_2)$. This proves the implication $T_1 \neq T_2 \Rightarrow \psi(T_1) \neq \psi(T_2)$.

(ii) This is proved similarly, using now part (ii) of Lemma 2.

(iii) Using the notation of Prop. 4 and assuming $T \subset V'$, $T \in \mathrm{Q}_n^{n-k}$, one can take $T = \mathrm{span}\{v'_{k+1}, \ldots, v'_n\}$. This being so, $U = T^\perp \cap V = \mathrm{span}\{v_1, \ldots, v_k\}$ and $U'$ can be identified with $\mathrm{span}\{v'_1, \ldots, v'_k\}$ so that $h|U \oplus U'$ is indeed non-degenerate. Since $\gamma(t)x = c(t)x$ for $t \in V'$, one obtains that $\varphi(T) = \wedge^+ U$ is a $2^{k-1}$-dimensional carrier space of a Weyl representation of $\mathrm{Spin}_{2k}$. Finally, if $x \in (\wedge^+ U)^\times$, then $N(x) = N_U(x) \oplus T$, which proves (27). □

Note that if one applies the definition of the map $\varphi$ to totally isotropic $(n-1)$-planes, such as $T_1 = \mathrm{span}\{v'_1, v'_3, v'_4, \ldots, v'_n\}$ and $T_2 = \mathrm{span}\{v'_2, v'_3, v'_4, \ldots, v'_n\}$, then one obtains $\varphi(T_1) = \varphi(T_2) = \mathbb{C}$, even though $T_1 \neq T_2$.

### *3.1. The Invariants*

We introduce now a simplified notation: we write $ax$ instead of $\gamma(a)x$ and if $A \subset \mathrm{Cliff}_{2n}$ and $x \in S$, then

$$Ax = \{ax \in S : a \in A\}.$$

**Theorem 3.** *The following three conditions on the spinor $x$ are equivalent:*

*(i) the nullity $\nu$ of $x$ is positive;*

*(ii) 0 is in the closure of the set $W_{\mathrm{unit}}x$, where $W_{\mathrm{unit}} = W \cap \mathrm{Pin}_m$ is the set of unit vectors;*

*(iii) the set $W_{\mathrm{unit}}x$ is not closed in $S$.*

*Proof.* To prove (iii)⇒(i), consider the linear map $f : W \to S$ given by $f(w) = wx$. Its kernel is $N(x)$. If $\nu = 0$, then $f$ is injective and the image by $f$ of the closed subset $W_{\mathrm{unit}}$ of $W$ is closed in $S$. The implication (ii)⇒(iii) is obvious because $0 \notin W_{\mathrm{unit}}x$. Finally, to prove (i)⇒(ii), let $\nu > 0$ so that there is $u \in N(x)^\times$ and one can find a vector $v \in W_{\mathrm{cone}}$ such that $uv + vu = 1$. One then has, for every $s \in \mathbb{R}^\times$, $q(s) \in W_{\mathrm{unit}}$, where $q(s) = \mathrm{e}^s u + \mathrm{e}^{-s} v$ so that $q(s)x = \mathrm{e}^{-s} vx$. The set $q(\mathbb{R}^\times)x$ contains 0 in its closure.

**Corollary 1.** *The set of all spinors of positive nullity is contained in the set of all spinors $x$ such that 0 is in the closure of* $\mathrm{Spin}_m x$.

Indeed, if the nullity of $x$ is positive, then 0 is in the closure of $(\mathrm{Pin}_m \cap W)x$ and, *a fortiori*, in the closure of the larger set $(\mathrm{Pin}_m \setminus \mathrm{Spin}_m)x = w.\mathrm{Spin}_m.x$, where $w \in W_{\mathrm{unit}}$. The map $S \to S$, given by $x \mapsto wx$, is a homeomorphism preserving the origin; if 0 belongs to the closure of $w.\mathrm{Spin}_m.x$, then it also belongs to the closure of $\mathrm{Spin}_m.x$.

A continuous function $J : S \to \mathbb{C}$ is an *invariant* of the action of the group $\mathrm{Spin}_m$ if, for every $x \in S$ and $a \in \mathrm{Spin}_m$, one has $J(ax) = J(x)$. For example, the scalar component $F_0$ of the quadratic covariant $F$ defined in §2.5 is an invariant.

**Corollary 2.** *If $x$ is a spinor of positive nullity and $J$ is an invariant, then* $J(x) = J(0)$.

*Proof.* This is a direct consequence of the preceding corollary. Explicitly, in the notation of the proof of Theorem 3, the map $\hat{q} : \mathbb{R} \to \mathrm{Spin}_m$, given by $\hat{q}(s) = q(s)(u + v) = \cosh s + (uv - vu)\sinh s$, defines a one-parameter subgroup of $\mathrm{Spin}_m$ and $\hat{q}(s)x = e^s x \to 0$ as $s \to -\infty$.

In particular, all the invariants formed from $F_k(x)$, $k = 0, \ldots, 2m$, by homogeneous tensor operations (products and contractions) vanish on spinors $x$ of positive nullity. There is no converse to Corollary 2: the Igusa invariant, which generates the algebra of invariants of the Weyl representation of $\mathrm{Spin}_{12}$, vanishes on the orbit of spinors of nullity 0, described in part (iv.c) of Proposition 6.

### *3.2. The Lacunae*

We are now ready to answer the following simple question: what are the possible values of the nullity of a Weyl spinor ? As a preliminary we have the following

**Lemma 3.** *Let $T \in \mathrm{Q}_n^k$, where $k < n$. The manifold*

$$X = \{U \in \mathrm{Q}_n^{k+1} : T \subset U\}$$

*has dimension* $2(n - k - 1)$.

*Proof.* Since $T \subset T^\perp$, one can find a space $T_1$ complementary to $T$ in $T^\perp$. Because of $k < n$, one has $T_1 \nsubseteq W_{\mathrm{cone}}$ and the intersection $T_1 \cap W_{\mathrm{cone}} \setminus \{0\}$ is a hypersurface in $T_1$ defined by the polynomial equation $h(w) = 0, w \in T_1^\times$. To specify $U$ appearing in the definition of $X$, it is enough to give the direction, dir $w$, of a vector $w \in T_1 \cap W_{\mathrm{cone}}$. Therefore, the manifold $X$ can be identified with $\mathrm{P}(T_1) \cap \mathrm{Q}_n^1$. Since $T_1$ has dimension $2(n - k)$, the manifold $X$ is of dimension $2(n - k - 1)$. □

To keep track of the dimensions, let us now denote by $S_n = (\wedge^+\mathbb{C}^n)^\times$ the space of non-zero Weyl spinors, of positive helicity, associated with the vector space $W = \mathbb{C}^{2n}$. We define

$$S_n^k = \{x \in S_n : \dim N(x) \geq k\},\ k = 0, 1, \ldots, n+1,$$

so that $S_n^{n+1} = \emptyset$ and

$$\Sigma_n^k = S_n^k \setminus S_n^{k+1},\ k = 0, 1, \ldots, n,$$

so that $S_n^n$ is the space of pure spinors, $\Sigma_n^k$ is the space of spinors of nullity $k$ and

$$S_n^n \subset S_n^{n-1} \subset \cdots \subset S_n^0 = S_n.$$

It is clear that, for every $a \in \mathrm{Spin}_{2n}$, one has $a\Sigma_n^k = \Sigma_n^k$, i.e. $\Sigma_n^k$ is a union of orbits of the group $\mathrm{Spin}_{2n}$. The set $\Sigma_n^k$ is either empty or open and dense in $S_n^k$. More precisely, we have

**Theorem 4.** *The set $\Sigma_n^{n-k}$, where $n - k \geq 0$ and $n = 1, 2, \ldots$, is empty if, and only if $k = 1, 2, 3$ or $5$.*

*Proof.* If $x \in S_n^{n-k}$, then there is $T \in \mathrm{Q}_n^{n-k}$ such that $x \in \varphi(T)$. Without loss of generality, one can take $T$ to be as in the proof of Theorem 2. Since, for $k = 1, 2$ and $3$, every Weyl spinor $x$ associated with the group $\mathrm{Spin}_{2k}$ is pure, one has, for these values of $k$, $\nu_U = k$ and (27) gives $\nu = n$: the same spinor $x$, considered relative to $\mathrm{Spin}_{2n}$ is also pure. This proves $S_n^{n-3} = S_n^{n-2} = S_n^{n-1} = S_n^n$. To show $S_n^{n-4} \neq S_n{}^n$ for $n \geq 4$, consider, in the notation of the proof of Theorem 2, the pure spinors $x = 1$ and $y = v_1 \wedge v_2 \wedge v_3 \wedge v_4$. Since $N(x) \cap N(y) = \mathrm{span}\{v_5', \ldots, v_n'\}$ is $(n-4)$-dimensional, on the basis of part (iii) of Prop. 5 one concludes that the spinor $x + y$ is not pure; its nullity is $n - 4$. Let now $x \in S_n^{n-5}$ and $T \in \mathrm{Q}_n^{n-5}$ be such that $x \in \varphi(T)$. According to part (iii) of Prop. 6, either $x$ is pure—and it then belongs to $S_n^n$ by a an argument similar to the previous one—or its nullity is 1. In the latter case, Eq. (27) gives $\nu = n - 4$. This proves $S_n^{n-5} = S_n^{n-4}$. Finally, to show that $\Sigma_n^k$ is non-empty for $n - k > 5$, consider $T \in \mathrm{Q}_n^k$, $k < n$. Let $X$ be the manifold defined in Lemma 3, and let $Y \to X$ be the vector bundle $Y = \{(x, U) : x \in \varphi(U),\ U \in X\}$. There is a tautological surjective map $Y \to S_n^{k+1} \cap \varphi(T)$; if $S_n^k = S_n^{k+1}$, then the map $Y \to S_n^k \cap \varphi(T) = \varphi(T)$ is also surjective and, therefore, $\dim Y = \dim X + \dim \varphi(U) \geq \dim \varphi(T)$. Using Lemma 3 and Theorem 2 one obtains the inequality

$$2(n-k-1) + 2^{n-k-2} \geq 2^{n-k-1} \quad \text{i.e.} \quad n - k \geq 1 + 2^{n-k-3}$$

which holds only for $n - k < 6$. Therefore, if $n - k > 5$, then $\Sigma_n^k \neq \emptyset$. □

### *3.3. The Dimensions*

Elementary arguments about dimensions, known already to Veblen and Givens [VG], have been at the origin of the notion of pure spinors: the projective space $\mathrm{P}(S)$ of directions of Weyl spinors is $(2^{n-1}-1)$-dimensional, whereas the manifold of all *mti* subspaces of $\mathbb{C}^{2n}$ has complex dimension $\frac{1}{2}n(n-1)$ (see [PeR, vol. 2, p. 453] for a simple proof of the last statement). For $n = 1, 2$ and 3, these dimensions coincide, but $2^{n-1}-1 > \frac{1}{2}n(n-1)$ for every $n > 3$. The following Lemmas will allow us to compute the dimensions of spaces of partially pure spinors of a given nullity.

**Lemma 4.** *The dimension of* $\mathrm{Q}_n^k$ *is* $2kn - \frac{1}{2}k(3k+1)$.

*Proof.* Let $W = \mathbb{C}^{2n}$; consider the tautological principal bundle

$$\mathrm{GL}_k \to \mathrm{E}_n^k \to \mathrm{Q}_n^k$$

such that

$$\mathrm{E}_n^k = \{(v_1,\dots,v_k) \in W^k : \mathrm{span}\{v_1,\dots,v_k\} \in \mathrm{Q}_n^k\}.$$

The dimensions of $\mathrm{E}_n^k$ and of its fibre being $2kn - \frac{1}{2}k(k+1)$ and $k^2$, respectively, one obtains $\dim \mathrm{Q}_n^k = \dim \mathrm{E}_n^k - \dim \mathrm{GL}_k = 2kn - \frac{1}{2}k(3k+1)$. □

**Lemma 5.** *Consider the bundle*

$$\Phi_n^k \to \mathrm{Q}_n^k \ \textit{such that}\ \Phi_n^k = \{(x,T) : x \in \varphi(T)\ \textit{and}\ T \in \mathrm{Q}_n^k\}. \tag{28}$$

*(i) The map* $\pi : \Phi_n^k \to S_n^k,\ (x,T) \mapsto x$ *is surjective.*
*(ii) If the set* $\Sigma_n^k$ *is not empty, then the map* $\tilde{\pi}$ *obtained by restricting* $\pi$ *to* $\pi^{-1}(\Sigma_n^k) \subset \Phi_n^k$ *is an injection into* $\Sigma_n^k$.

*Proof.* (i) The map $\pi$ is surjective because, if $x \in S_n^k$, then $\dim N(x) \geq k$, one can choose $T \subset N(x)$ of dimension $k$ and then $(x,T) \in \Phi_n^k$. A fibre $\varphi(T)$ of the bundle (28) is mapped by $\pi$ injectively into $S_n^k$. To prove (ii), suppose that $\Sigma_n^k \neq \emptyset$ and $\tilde{\pi}$ is not injective. Let $T_1, T_2 \in \mathrm{Q}_n^k$, and $x$ be such that $x \in \varphi(T_1) \cap \varphi(T_2) \cap \Sigma_n^k$. This implies $T_1 \subset N(x)$, $T_2 \subset N(x)$ and, therefore, $T_1 + T_2 \subset N(x)$. If $T_1 \neq T_2$, then $\dim(T_1+T_2) > k$ and this contradicts $x \in \Sigma_n^k$. □

**Corollary 3.** *If* $\Sigma_n^k$ *is not empty, then its dimension equals that of* $\pi^{-1}(\Sigma_n^k)$.

**Theorem 5.** *In the notation of* §*3.2 one has*
*(i)* $\dim \Sigma_n^n = 1 + \frac{1}{2}n(n-1)$,
*(ii)* $\Sigma_n^{n-1}, \Sigma_n^{n-2}, \Sigma_n^{n-3}$ and $\Sigma_n^{n-5}$ are empty,
*(iii)* $\dim \Sigma_n^k = k(2n - \frac{1}{2}(3k+1)) + 2^{n-k-1}$ for $k = n-4$ *and* $k < n-5$.

*In particular, for $n > 5$, the set $\Sigma_n^0$ is open and dense in $S$: a Weyl spinor in a general position is not annihilated by any non-zero vector.*

*Proof.* Part (ii) follows from Theorem 4; to prove (i) and (iii), assume $\Sigma_n^k \neq \emptyset$, so that Lemma 5 and its Corollary can be applied to give

$$\dim \Sigma_n^k = \dim \pi^{-1}(\Sigma_n^k) = \dim \Phi_n^k = \dim \mathrm{Q}_n^k + \dim \varphi(T),$$

where $T \in \mathrm{Q}_n^k$ so that, according to Theorem 2, one has $\dim \varphi(T) = 2^{n-k-1}$ for $k = 1, \ldots . n-1$ and $\dim \varphi(T) = 1$ for $T \in \mathrm{Q}_n^{n+}$. It now suffices to use Lemma 4 to obtain the announced dimensions. □

The classification of spinors according to their nullity is coarse in the sense that, with the exception of the orbits of pure spinors $\Sigma_n^n$, and a few others, the strata $\Sigma_n^k$ are collections of many orbits. This coarse classification has been presented here for arbitrary $n$, whereas the precise classification, along the lines developed by Igusa, is limited to $n < 8$; in the words of Popov: "the case we are investigating is one of the last where the problem of classifying spinors has a reasonable meaning and can be conclusively solved" [Pp, p. 182]. It is worth noting that there are two 'dimensional thresholds' in the study of spinor representations: the first occurs at dimension $m = 6$ of the underlying vector space $W$. For $m > 6$, the dimension of the manifold of all *mti* subspaces of $W$ is smaller than that of the (projective) space of spinors. The second is at $m = 14$: for $m > 14$, the dimension of the space of spinors is larger than that of the group $\mathrm{Spin}_m$.

## An explanatory note and acknowledgments

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I am responsible for this presentation of our joint work and its shortcomings. *A.T.*

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